

Two-dimensional Schrödinger operators with fast decaying potential and multidimensional L_2 -kernel*

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In this note using Moutard transformations we show how explicit examples of two-dimensional Schrödinger operators $L = -\Delta + u(x, y)$ with fast decaying potential and multidimensional L_2 -kernel may be constructed. In the explicit examples below the potential $u(x, y)$ and square-summable solutions $\psi(x, y)$ of the equation $L\psi = 0$ are smooth rational functions of x and y . In the best case constructed below, the potential u and the eigenfunctions ψ decay as $1/r^8$ and $1/r^3$ respectively, where $r^2 = x^2 + y^2$. The potentials constructed are exactly solvable at the zero energy level, in the sense that all solutions of $L\psi = 0$ can be built from harmonic functions and their construction requires only quadratures.

For operators with so fast decaying potentials a suitable spectral theory is known ([1, 2]). In dimension one existence of square-summable eigenfunctions at zero energy level is impossible. The inverse spectral problem for multidimensional Schrödinger operators at a fixed energy level was considered in [3] for the first time, where the case of double periodic potential was investigated. For fast decaying potentials this problem was studied mostly at positive energy levels [4], or below the ground state [5].

The Moutard transformation for the two-dimensional Schrödinger operator

$$L = -\Delta + u = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + u(x, y)$$

produces another Schrödinger operator \tilde{L} with the potential \tilde{u} :

$$\tilde{L} = -\Delta + \tilde{u} = -\Delta + (u - 2\Delta \log \omega),$$

where ω is a solution of the equation $L\omega = 0$. From this we have $\tilde{u} = 2\frac{\omega_x^2 + \omega_y^2}{\omega^2} - u$. For every solution φ of the equation $L\varphi = 0$ one can construct a solution $\tilde{\varphi}$ of

*The authors acknowledges partial financial support from RFBR (grants 06-01-00094a, 06-01-00814a), complex integration project 2.15 SB RAS Max Plank Mathematical Institute (I.A.T.) and DFG Research Unit 565 "Polyhedral Surfaces" (TU-Berlin, S.P.Ts.)

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the equation $\tilde{L}\tilde{\varphi} = 0$:

$$(\omega\tilde{\varphi})_x = -\omega^2 \left(\frac{\varphi}{\omega} \right)_y, \quad (\omega\tilde{\varphi})_y = \omega^2 \left(\frac{\varphi}{\omega} \right)_x.$$

The construction of $\tilde{\varphi}$ from φ is reduced to quadratures and the constructed function $\tilde{\varphi}$ is uniquely determined up to terms of the form $C\omega^{-1}$, where $C = \text{const}$. Keeping in mind this ambiguity we will denote hereafter the result of the Moutard transformation as M_ω : $\tilde{u} = M_\omega(u)$, $\tilde{\varphi} = M_\omega(\varphi)$. Note that $\tilde{L}\omega^{-1} = 0$.

For the case of a one-dimensional potential $u = u(x)$ and $\omega(x, y) = f(x)e^{ky}$ one has $L = -\frac{\partial^2}{\partial y^2} + A^\top A + k^2$, where $A = -\frac{\partial}{\partial x} + \frac{f_x}{f}$, $A^\top = \frac{\partial}{\partial x} + \frac{f_x}{f}$, so the Moutard transformation will be reduced to the Darboux transformation of the one-dimensional operator $L' = -\frac{\partial^2}{\partial x^2} + (u + k^2)$, namely $L \rightarrow \tilde{L} = -\frac{\partial^2}{\partial y^2} + AA^\top + k^2$. Iterations of the Darboux transformation starting from the potential $u_0 = 0$ and $f(x) = x$ give all exactly solvable one-dimensional rational potentials [6]. Note that all these potentials are singular.

GENERAL CONSTRUCTION SCHEME. Take two Moutard transformations of the Schrödinger operator with some potential u_0 defined by the solutions ω_1 and ω_2 and denote the resulting potentials u_1 and u_2 respectively. The transformed functions $M_{\omega_1}(\omega_2)$ are defined up to terms C/ω_1 and satisfy the equation $(-\Delta + u_1)\psi = 0$. The same holds for the transformed functions $M_{\omega_2}(\omega_1)$. Fix a function θ_1 from the set $M_{\omega_1}(\omega_2)$ and take the function $\theta_2 = -\frac{\omega_1}{\omega_2}\theta_1$ from the set $M_{\omega_2}(\omega_1)$. Now apply to the operator $(-\Delta + u_1)$ the Moutard transformation defined by the function θ_1 and similarly the Moutard transformation defined by θ_2 to the operator $(-\Delta + u_2)$. The following result is well known:

Lemma 1 $M_{\theta_1}(u_1) = M_{\theta_2}(u_2) = u$. The functions $\psi_1 = \frac{1}{\theta_1}$ and $\psi_2 = \frac{1}{\theta_2}$ satisfy the equation $L\psi = 0$, where $L = -\Delta + u$.

In this construction we have a free scalar parameter for the choice of the function θ_1 from the set $M_{\omega_1}(\omega_2)$. This parameter can be used in some cases to build a nonsingular potential u and the functions ψ_1 and ψ_2 .

EXAMPLE 1. Let $u_0 = 0$, $\omega_1 = x + 2(x^2 - y^2) + xy$ and $\omega_2 = x + y + \frac{3}{2}(x^2 - y^2) + 5xy$. Choosing θ_1 appropriately (we omit its lengthy formula) one gets

$$u = -\frac{5120(1 + 8x + 2y + 17x^2 + 17y^2)}{(160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + 16xy^2 + 4y^3 + 17(x^2 + y^2)^2)^2}, \quad (1)$$

$$\begin{aligned} \psi_1 &= \frac{x + 2x^2 + xy - 2y^2}{160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + 16xy^2 + 4y^3 + 17(x^2 + y^2)^2}, \\ \psi_2 &= \frac{2x + 2y + 3x^2 + 10xy - 3y^2}{160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + 16xy^2 + 4y^3 + 17(x^2 + y^2)^2}. \end{aligned} \quad (2)$$

Theorem 1 The potential u given by (1) is smooth, rational and decays as $1/r^6$ for $r \rightarrow \infty$.

Functions ψ_1 and ψ_2 given by (2) are smooth, rational, decay as $1/r^2$ for $r \rightarrow \infty$ and span a two-dimensional space in the kernel of the operator $L = -\Delta + u : L_2(\mathbb{R}^2) \rightarrow L_2(\mathbb{R}^2)$.

EXAMPLE 2. Let $u_0 = 0$, $\omega_1 = x + \frac{x^2 - y^2 - 3xy}{5} + 2(-x^3 - 3x^2y + 3xy^2 + y^3)$ and $\omega_2 = x + y + \frac{x^2 - y^2}{2} - \frac{xy}{5} - 4(3x^2y - y^3)$. For some appropriate θ_1 one obtains a smooth rational potential u decaying as $1/r^8$, as well as smooth rational ψ_1 and ψ_2 in the L_2 -kernel of the Schrödinger operator with the potential u , ψ_1 and ψ_2 decay as $1/r^3$. We omit here the explicit formulas.

We conjecture that increasing the degree of the initial harmonic polynomials ω_1 and ω_2 one can for every $N > 0$ construct potentials u and their eigenfunctions ψ_1 and ψ_2 decaying faster than $1/r^N$.

The authors would like to thank P.G.Grinevich and S.P.Novikov for useful discussions.

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